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Mathematics: Method Without Metaphysics

Elaine Landry*

Department of Philosophy, University of California, Davis,
Davis, California 95616-8673, U.S.A.

ABSTRACT

I use my reading of Plato to develop what I call as-ifism, the view that, in mathematics, we treat our hypotheses as if they were first principles and we do this with the purpose of solving mathematical problems. I then extend this view to modern mathematics showing that when we shift our focus from the method of philosophy to the method of mathematics, we see that an as-if methodological interpretation of mathematical structuralism can be used to provide an account of the practice and the applicability of mathematics while avoiding the conflation of metaphysical considerations with mathematical ones.

1. EXACTNESS

In this paper I will answer three questions: Wherein lies the exactness of mathematics?; Wherein lie the conditions for speaking about mathematical objects?; and, Wherein lies the exactness of the exact sciences?. Along the way, I carve out an *as-if* interpretation of mathematical structuralism by disentangling methodological considerations from metaphysical ones. I begin first with Plato and draw important lessons from his account of mathematics. More specifically, my aim will be to show that much philosophical milk has been spilt owing to our confusing the method of mathematics with the method of philosophy, and that, as a result, mathematical considerations are conflated with metaphysical ones. To this end, I use my reading of Plato to develop what I call *as-ifism*, the view that, in mathematics, we treat our hypotheses *as if* they were first principles *and* we do this with the purpose of solving mathematical problems. I then extend this view to modern mathematics wherein the method of mathematics becomes the axiomatic method, noting that this engenders a shift from as-if hypotheses to as-if axioms and a shift from the investigation of kinds of

*http://orcid.org/0000-0001-7820-7974; E-mail:emlandry@ucdavis.edu

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objects to the consideration of systems that have a structure, so that objects are nothing but positions in a structure. This structuralist perspective is then set within a Plato-inspired *methodological* context to argue for an *as-if* interpretation of mathematical structuralism. I first contrast structural as-ifism with structural *if-thenism* and, along the way, I pause to note the way in which the confusion of the method of mathematics with the method of philosophy, witnessed well by the Frege–Hilbert debate, has led to the continued conflation of mathematics with metaphysics. Finally, I combine my *as-if* account of applicability with Maddy's more recent *enhanced if-thenist* approach to show how such conflations can and should be avoided, especially in light of current confusions amongst structural realists.[1] My overall lesson is this: when we shift our focus from the method of philosophy to the method of mathematics, we see that an *as-if methodological* interpretation of mathematical structuralism can be used to provide an account of the *practice* and the *applicability* of mathematics while avoiding the conflation of metaphysical considerations with mathematical ones.

2. PLATO

In this section I aim to show that, contrary to popular interpretations, Plato kept a clear distinction between mathematics and metaphysics, and the knife he used to slice the difference between the two was method. In sum, I will show that Plato answered the above questions as follows: The exactness of mathematics lies in the *precision* of its *definitions* and the *stability* of its *method*; the object-level conditions for speaking about mathematical 'objects themselves' are found in pure mathematical theories and the meta-level conditions for speaking about mathematical 'kinds of objects' (*i.e.*, what all mathematical objects themselves have in common) are found in the pure theory of geometrical proportion; and, the exactness of the exact sciences lies in the *applicability* of object-level *pure* mathematics.

Early in his philosophical development Plato realized that mathematics had something to offer the philosopher in his quest for knowledge. The two aspects of mathematics that Plato finds particularly helpful are the precision of its definitions and the stability of its method. The epistemological value of both of these is never better demonstrated than in the *Meno*. When confronted with Meno's ever changing definition of virtue in terms of its instances, Plato says he wants a definition, like his mathematical definition of shape, that covers all instances, without assuming any of the terms of the definition as known. Just after this request Meno confronts Plato with his famous epistemological

[1] Structural realism arises from a no-miracles argument aimed at structure, as opposed to traditional scientific realism which aims its no-miracles argument at objects and, in so doing, falls victim to the pessimistic meta-induction argument. More specifically, structural realism is the view that, given the continuity of mathematical structure over successive empirically successful scientific theories, we should be realists about the mathematical structure, as opposed to the ontology, of our successful scientific theories. It typically comes in two versions: for example, Worrall's [1989] *epistemic* version (ESR) holds that all we *know* about the world is its mathematical structure; in contrast, French's [2012] ontic version (OSR) holds that all *there is* of the world is mathematical structure.

paradox: how will you look for something when you don't in the least know what it is, how will you set up something as the object of your search when you don't know the object of your search. Plato here realizes that if he is to move past the problems associated with Socrates' elenctic method, he needs a new method. This new method is the mathematician's *hypothetical method*.[2] This is the method wherein we begin our search with a hypothetical definition of the object of our search and our task is then to determine if the answers that we seek in our attempt to solve a problem can be derived from this definition.

For example, in solving the Meno problem, I treat the length of the line that doubles the area of a two-unit square *as if* it were a stable object, but it is not. It is only because of the precision of the definitions of square and of diagonal, together with the assumed truth of the Pythagorean theorem, that I can reason down to the conclusion that the length will be the length of the diagonal of the two-unit square. But I cannot know the length of this line as a stable object since the length is $2\sqrt{2}$. The diagonal, then, is the definition that yields the needed hypothesized object from which our answer can be derived, again, without our having to know what that object itself is. That is, in reply to Meno's paradox, they proceed to solve their mathematical problem by constructing the square whose length is the given by the length of the diagonal, and this even though they cannot know what the length of that side is.[3] Likewise, to solve the problem of the *Meno*, *i.e.*, whether virtue is teachable, does it come by practice or is it a natural aptitude, Socrates requests of Meno that they 'make use of a hypothesis — the sort of thing ... that geometers often use in their inquiries' [Plato, 1956, 86e]. In light of his just-demonstrated resolution of Meno's paradox, Socrates now replies 'Let us do the same about virtue. Since we don't know what it is ... let us use a hypothesis in investigating whether it is teachable or not [87b], and so they begin with the hypothesis that virtue is knowledge. The problem is that while in the mathematical case, the use of the hypothetical method can yield 'a knowledge on the subject as accurate any anybody's' [85c–85d], in the philosophical case, the use of the hypothetical method can only yield 'true opinion' [1956, 97b], and this is because the hypotheses themselves, like the statues of Daedalus, are not 'tied down' [97d], *i.e.*, they are not first principles. 'Once they are tied down, they become knowledge and are stable. That is why knowledge is something more valuable than right opinion. What distinguishes one from the other is the tether (you tether them by working out the reason)' [98a]. But Plato is here quick to note that 'for *practical purposes* right opinion is no less useful than knowledge, and the man who has it is no less useful than the one who knows' [98c; emphasis added].

[2] For further details see my [2012] argument that the method of mathematics, as presented in the *Meno*, is not recollection but rather is the hypothetical method.

[3] Plato, unlike the Pythagoreans, accepted irrational numbers as long as an account could be given for them. That is, we are here presented with a 'proof' that $2\sqrt{2}$ can be accounted for as a geometric measure, *viz.*, as the length of the diagonal of a two-unit square. By taking numbers as geometric measures, as opposed to Pythagorean arithmetical units, Plato here made a distinction between irrational numbers that have an account (*logos*) and those that do not (*alogos*).

The question that remains for us to consider is: Does the mathematician, like the philosopher, have to tether his hypotheses? In answer to this question, we turn to Plato's *Republic* where the distinctions between the mathematician's and the philosopher's methods, and between mathematical knowledge and philosophical knowledge, are further developed. More pointedly, here we are explicitly told the sense in which the mathematician's hypothetical method is *distinct from* the philosopher's dialectical method. The mathematician's hypothetical method *begins with* hypotheses taken *as if* they were first principles, *i.e.*,

> students of geometry, calculation, and the like *hypothesize* the odd and the even, the various figures, the three kinds of angles, and other things akin to these in each of their investigations, *regarding them as known*. These they treat as [absolute] hypotheses and *do not think it necessary to give any argument for [account of] them*, either to themselves or to others, *as if* they were evident to everyone. And going from these first principles through the remaining steps, [They take their start from these, and pursuing the inquiry from this point on *consistently*] they arrive [conclude], in full agreement at *the point they set out to reach in their investigation*. [Plato, 2005, 510c–510d, emphasis added]

The mathematician's hypotheses are thus taken *as if* they were first principles, but the mathematician understands that they are not. Indeed, as noted in the last sentence, the purpose of the mathematicians' *as-if* hypotheses is *practical*, that is, its use is aimed only at solving a given mathematical problem. To this practical purpose, the hypothetical method allows us to *reason down* from a hypothesis to a conclusion; *internal consistency* then tells us what is *possible/impossible* in the context of a given problem. Because the mathematician takes his hypotheses *as if* they were first principles, all the while realizing that they are not, he thus uses *thought* (as opposed to understanding) and his process yields *a kind* of knowledge or *true opinion* (as opposed to knowledge *itself*). Thus, unlike the philosophers' method, the mathematicians' method is not aimed at knowledge itself.

The philosopher's *dialectical method*, in contrast, in so far as it aims at knowledge itself and not just true opinion, must begin with a hypothesis *qua* hypothesis and so must *reason up* towards an *un*hypothetical first principle. These first principles, unlike hypotheses, must be tethered, and thus philosophical knowledge requires a stable domain of objects (objects unlike, for example, the statues of Daedalus) to tether to, or justify, or 'tie down' hypotheses as first principles; that is, it requires an ontology of Forms. Once so tethered, the philosopher may then reason down from a Form-fixed first principle to a conclusion; *external consistency*[4] then tells us what *must be the case* given the

[4] The difference between internal and external consistency can be explained as follows: internal consistency is measured against what one hypothesizes *as if* it were true, *e.g.*,

unchanging nature of the Forms. As such the dialectic method, in so far as it alone uses *reason itself*, yields *knowledge itself*.

> Also understand, then, that by the other subsection of the intelligible I mean what *reason itself* grasps by the power of *dialectical discussion*, treating its hypotheses, not as first principles [absolute beginnings], but as *genuine hypotheses* (that is, as stepping stones and links in a chain), in order to arrive at what is *unhypothetical* and *the first principle* of everything. Having grasped this principle, it reverses itself and, keeping hold of what follows from it, comes down to a conclusion, making no use of anything visible at all, but only of *forms themselves*, moving on through forms to forms, and ending in forms. [Plato, 2005, 511b–511c; emphasis added]

What is important to note is that, because the method of mathematics *begins with* hypotheses and treats them *as if* they were first principles, mathematics *does not need* a metaphysics or foundational ontology of Forms, or of geometrical objects (see [Tait, 2002]), that fixes its hypotheses as first principles and, in so doing, provides an account of what the objects of mathematics are. The method of mathematics is *distinct from* the method of philosophy, and as such so is its epistemology *and* its ontology. The mathematical method yields *a kind* of knowledge or true opinions that are 'reliable guides to solving problems' because they are born out of precise definitions and a stable method, but this method cannot yield knowledge itself. Only the philosophical method yields knowledge itself; that is, yields beliefs that are both stable *and* fixed; its hypotheses are taken as hypotheses and reason itself is further employed to tether these to a fixed domain of objects, yielding knowledge based on unhypothetical first principles. Thus, the exactness of philosophy as a science is found in the fixity of its objects, *i.e.*, in Forms, and the stability of the dialectical method. In contrast, and in answer to our first question, the exactness of the mathematics as a science is found in the precision of its definitions and the stability of the hypothetical method. Even Glaucon is shocked to hear that philosophy as a science is more exact (clearer) than is mathematics; that, because of their differences in method, mathematics yields *a kind* of knowledge, knowledge based on thought, whereas philosophy yields knowledge *itself*, *i.e.*, knowledge based on understanding.

> I understand, though not adequately — you see, in my opinion, you are speaking of an enormous task. You want to distinguish *the part of what*

the claim that the length of the side that doubles the area of the two-unit square is $2\sqrt{2}$ is measured against the hypothesis that we treat the length as if it was a diagonal; and, the claim that virtue is teachable is measured against the hypotheses that virtue is knowledge. External consistency, in contrast, is measured against what is true as fixed by Forms themselves, *e.g.*, the claim that virtue is good is measured against the form Virtue participating in the form Good.

> *is* [this aspect of reality] and what is intelligible, the part looked at by the science of *dialectical discussion, as clearer than the part* [*as something truer and more exact than the objects*] looked at by the so-called sciences — those for which *hypotheses are first principles* [assumptions are *arbitrary* starting points]. And although those who look at the latter part are forced to do so by means of *thought* rather than *sense perception*, still, because they do not go back to a genuine first principle in considering it, but proceed from hypotheses, *you do not think that they have true understanding* of them, even though — given such a first principle — they are intelligible. And you seem to me to call the state of mind of the geometers — and the others of that sort — *thought* but *not understanding*; thought being *intermediate between belief and understanding.* [Plato, 2005, 511c–511d; emphasis added]

We now turn to the second question, *viz.*, Wherein lie the conditions for speaking about mathematical objects themselves? Whatever we take the Forms to be, the conditions for speaking about philosophical kinds of objects, like virtue, justice, *etc.*, are to be found in a metaphysical theory, that is, in their first principles being or being tethered to Forms. In context of mathematics, the object-level conditions for speaking about mathematical objects, like numbers themselves, squares themselves, are found in a mathematical theory, *viz.*, in arithmetic and geometry, respectively. Plato, also in the *Republic*'s Divided Line, further takes time to order the various mathematical theories, and the learning of them, as follows: arithmetic, geometry, plane geometry, spherical geometry and, finally, the geometrical[5] theory of proportion. The theory of proportion, however, is further taken as having a double role to play: it serves both as a mathematical theory and as providing a *meta-level* or over-arching account of what all the other mathematical 'kinds of objects' have in common.

The theory of proportion is thus used to organize what we say about mathematical objects as geometrical ratios; it gives us a *meta-level* account of mathematical kinds of objects without our needing a fixed domain of mathematical Forms. That is, it allows for the 'investigation of *all* the [mathematical] subjects we have mentioned' with the aim of arriving at '*what they share in common* with one another and what their affinities are, and *drawing conclusions about their kinship*' [Plato, 2005, 531d, emphasis added]. This dual role of the theory of proportion allows us to solve both mathematical and meta-mathematical problems. It solves the *mathematical problem* of how to account for irrational numbers, numbers like $2\sqrt{2}$, by taking numbers themselves as a proportioned kind of objects, *i.e.*, as measures of geometrical ratios. And

[5] There are two competing theories of proportion at play here: the arithmetical theory of the Pythagorean Archytas and the geometrical theory of Theodorus and Theaetetus. Plato advocates for the geometrical because, for example, only this would allow for the inclusion of irrational numbers that have a logos when they are taken *as if* they are geometrical measures. See [Fowler, 2003] for more on this distinction and how it plays out in Plato's account of mathematics.

it solves the *meta-mathematical problem* of what, if not mathematical Forms, provides an over-arching account of mathematical kinds of objects. Thus, akin to the metaphysical role that the theory of Forms plays, the theory of proportion plays the role of providing the meta-level conditions for speaking about mathematical objects as the same kinds of objects.

Finally we turn to question three, *viz.*, Wherein lies the exactness of the exact sciences? The reply that Plato offers is that the exactness of the exact sciences, like cosmology and astronomy, lies in the *applicability* of *pure* mathematics; it does not lie in any empirically motivated or empirically interpreted mathematics. In modern language, it is pure mathematics that accounts for the exactness of the exact sciences by providing abstract models, so that, as Maddy states, 'the applied mathematician's claim [is] that this abstract model resembles the worldly situation well enough to be used for the purposes [of solving physical problems]' [2022, pp. 270–271]. Plato, likewise, makes the distinction between empirical and pure mathematics, and this for the same purpose of accounting for its applicability. In the Divided Line, he first takes great pains to note that 'no one uses it [mathematics] correctly' [2005, 523a] and this because there is a distinction that must be made between empirical mathematics and pure mathematics, *e.g.*, between the counting numbers of 'tradesmen and retailers' [525c] and numbers themselves, between those 'accounts of its practitioners' ... [that] talk of squaring' [527a] and squares themselves. He then goes on to use this distinction to note that in the exact sciences, *i.e.*, those sciences like astronomy and cosmology, where we aim to *apply* mathematics, we are to use pure mathematics and *not* any empirically motivated mathematics. In astronomy, for example, we are to look to the pure mathematics of spherical geometry in motion; we are not to look to the empirical mathematics motivated by motions of the 'ornaments of the heavens' [529b]. Likewise, in cosmology, we are to look to the pure mathematics of geometric proportions; we are not to look to the empirical mathematics motivated by arithmetic ratios of 'audible concordances' [531c]. We may use these empirically constructed 'models to help us study these other things' [529e], but 'we will leave the things in the heavens alone' and so 'just as in geometry, then, it is by making use of problems that we will pursue' our account of the exact sciences [530b]. Thus, the exactness of exact science lies in its use of pure mathematics to solve physical problems.

As I have shown, Plato's account of the exactness of the mathematics as a science is found in the precision of its definitions and the stability of the hypothetical method; it is not found in the fixity of any objects that tether its hypotheses as unhypothetical first principles. Moreover, his account of the exactness of the exact sciences lies in the applicability of a pure mathematical theory, not in any empirically motivated or empirically interpreted mathematical theory. The pure theory of proportion, for example, serves as both a science for mathematics itself, by using geometric ratios to organize what we say about mathematical kinds of objects meta-mathematically, and as a science for cosmology, by using geometric ratios to organize what we say about the good order, or the harmony, of the cosmos.

Having now answered our three questions, let us pause to sum up what I will call Plato's *methodological as-ifism.* The mathematician's hypotheses are taken *as if* they were first principles, but they are not. The purpose of the mathematician's method is to solve a given mathematical problem, and *it is in virtue of this methodological use* that we are justified in taking our hypotheses *as if* they were first principles. Mathematical hypotheses, as distinct from philosophical hypotheses, are thus not to be justified by tethering them to a domain of fixed objects or Forms. Yet, these differences in method demand differences in both epistemology and ontology. The mathematical method yields *a kind* of knowledge, that is, yields beliefs that are 'reliable guides to solving problems' [532b] because they are born out of stable definitions and a reliable method. Mathematical objects are objects of thought (conjecture),[6] but not objects of understanding; they are not as real as philosophical objects, but, as objects of thought, unlike objects of imagination, they are still 'concerned with being' [534a]. Only the dialectical method of philosophy yields *knowledge itself*; yields true beliefs that are themselves further fixed to, or tethered by, a domain of fixed objects, that is, tethered to Forms as objects of understanding. Thus, *against metaphysical realism*, mathematics does not need a metaphysics of Forms that fixes its hypotheses as unhypothetical first principles and in so doing accounts for, or tethers, the truth of its hypotheses. This is because, while philosophy as a science is founded on the dialectical method and the stability of its metaphysical objects, mathematics as a science is founded on the hypothetical method and the stability of its definitions.

This point marks the current confusion amongst both *structuralist philosophers of mathematics* and *structuralist philosophers of science*: they continue to conflate the hypothetical method of mathematics with the metaphysical method of philosophy. The correction I want to make is as follows: when I say that a mathematical object exists what I mean is that, in my aim of solving a mathematical problem, I treat my hypothesis *as if* it were a true first principle and, in doing so, I act *as if* it were tethered to an object. I recognize, however, that my hypothesis is not a first principle, and so my object is taken as an object of thought (conjecture) and not an object of understanding. Thus, we come to what I will call *methodological as-if realism*: in mathematics, we treat our hypotheses *as if* they were true first principles, and *consequently*, we treat our objects *as if* they exist, and we do this for the purpose of solving a mathematical, a foundational, or a physical problem. As-if mathematical realism is thus *distinct* from metaphysical realism: for the as-if

[6] Note that, at 511d–511e, the term that Plato uses for 'imagination' is *eikasai.* This word is Plato's own creation, some translate it as 'imagination', as derived from *eikon* or 'imagine', and others as 'conjecture', as derived from *eikaz* (*estahi*) or 'to guess at'. He uses this term again at 534a in his claim that the mathematician's faculty of thought is *akin to* the faculty of imagination, thus it seems clear that we are to take objects of thought as akin to the objects of imagination; both being objects of conjecture.

mathematical realist, *existence is a consequence of truth*, while for the metaphysical realist, truth is a consequence of existence. My aim now is to extend my methodological as-ifism to modern mathematics, wherein the method of mathematics becomes the axiomatic method, and to see if we can use this correction to clear the confusions in current structuralist interpretations of mathematics.

3. THE AXIOMATIC METHOD

The nineteenth century view of mathematics saw the exactness of mathematics as lying in the precision of its definitions and the stability of the, now characterized, *axiomatic method*. Aiming to connect this view to my methodological as-ifism, two issues remain to be discussed: the first is what accounts for the precision of mathematical definitions; and, the second is whether this method is still to be taken as hypothetical in nature. To investigate the first, I will examine the Frege–Hilbert debate over the nature of both definitions and axioms, and the relation between the two. In consideration of the second, I will examine the if-thenism of both Frege and Hilbert and see how it lines up against Plato's as-ifism. As we will see, both of these issues are intertwined.

As is well known, Frege and Hilbert disagreed over the nature of axioms. What I want to point out here, however, is that Frege is a prime example of someone who confused the method of mathematics with the method of philosophy, that is, he saw mathematical axioms as *truths* fixed over a *stable domain* of objects; so he thought he needed a background foundation to yield a first-principled account of the truth of axioms in terms of such objects. In the case of arithmetic, whether one takes this as the need for set theory as a background foundation providing a first-principled account of objects in terms of extensions, or as the need for logic as a foundation providing a first-principled account of objects as logical objects, what is clear is that a meta-mathematical domain of existing objects was needed to fix the *intended interpretation* of arithmetical axioms as truths. For the Fregean axioms-as-first-principles account, the primitive terms employed by the axioms must be defined over a stable domain *before* the statement of the axioms. That is, these definitions must be logically constructed in the case of arithmetic and constructed on the basis of the Kantian pure intuition of space in the case of geometry. In contrast, Hilbert took axioms as implicit definitions over a *variable* domain, so that axiom systems themselves are but *schemata* for implicitly defining those concepts expressed as the primitive terms that are then *variously interpreted* by the objects that satisfy the axioms.

For Frege the precision of mathematical definitions was to be justified by assuming the *truth* of the axioms, again, truth as fixed logically, in the case of arithmetic, or truth as fixed philosophically by Kantian intuition, in the case of geometry. That is, Frege's *meta-mathematical* account of the *method* of mathematics was: *if* the axioms are true, *then* this theorem can be justified. For Hilbert, however, the precision of definitions was justified by assuming the *consistency* of the axioms; hence, Hilbert's famous quote: 'if the arbitrary postulated axioms do not contradict each other with their collective consequences, then they are true, and the things defined by means of the axioms exist. That for me, is the criterion of truth and existence' [1899]. So, Hilbert's *meta-mathematical* account of the *method* of mathematics was: *if* the axioms are consistent, *then* this theorem can be taken as true.[7] While they disagreed about the relation between definitions and axioms, in answer to our first question, for both Frege and for Hilbert the exactness of mathematics was taken to arise from the precision of its definitions and the stability of the axiomatic method.

The question remaining is: What is to guarantee the *stability* of the axiomatic method? If it is logic, then does Frege's and Hilbert's meta-mathematical if-thenist account collapse to formalism or deductivism? Both thought it did. More to the point, fearing formalism, both Frege and Hilbert, came to reject *logical* if-thenism as a meta-mathematical account of the stability of method of mathematics. What I will now consider is whether these *logically interpreted if-thenist* views of the stability of the axiomatic method can be weakened to the *methodologically interpreted as-if* view that Plato seems to be offering up. As detailed by Resnik [1980], Frege developed two forms of if-thenism. According to the first, what I will call, the *deductive if-thenist*, option 'mathematics is in the business of establishing results in pure logic'. Presuming, then, that '$\mathbf{A}$ stands for a quantificational schema diagramming the supposed axioms and $\mathbf{T}$ stands for a quantificational schema diagramming the supposed theorem of the theory' [1980, p. 117], the first option can either be schematically expressed as '$\mathbf{A} \supset \mathbf{T}$' is logically valid (logically provable) or as the claim that $\mathbf{T}$ is a logical consequence of $\mathbf{A}$ ($\mathbf{T}$ is logically derivable from $\mathbf{A}$). Resnik next notes that Frege could not have accepted the latter approach since for Frege only interpreted, and therefore meaningful, sentences (as opposed to quantificational schemata) can have logical consequences.

On the second option, Frege 'views a mathematical theory as studying the properties of all structures satisfying certain defining conditions, but he never makes use of the assumption that such structures exist' [Resnik, 1980, p. 117]. Schematically, this second option can be expressed as '$\mathbf{A} \models \mathbf{T}$' ($\mathbf{T}$ is semantically entailed by $\mathbf{A}$).[8] Speaking to the virtues of this second, what I will call the *structural if-thenist*, option Resnik notes that such a view, 'can rid mathematics of ontological presuppositions while retaining its apparent descriptive character ... [and reduce] the epistemology of mathematics to that of logic, and account for the centrality of proof in mathematics' [1980, p. 118]. Noting yet another virtue, Resnik further claims that this structuralist option offers a straightforward account of *applicability*, that is, 'when one finds a physical

[7] Implicit in Hilbert's view is what is now known as the completeness theorem, *viz.*, that every first-order expressible consistent set of sentences is satisfiable. See Section 5 for more on the role of the completeness theorem in our meta-mathematical account of satisfiability.

[8] For my purposes, I will read semantic entailment in terms of satisfiability as follows: For any interpretation i, if i satisfies $\mathbf{A}$, then i satisfies $\mathbf{T}$.

structure satisfying the axioms of a mathematical theory, the application of that theory is immediate'. And lastly, Resnik notes a final virtue, *viz.*, that such a structural if-thenist approach is in line with the development of abstract structures, like group theory and topology, and, I would add, category theory.

Despite all of these virtues, there are problems, and, if Resnik is to be believed, they are problems that threatened to topple the structural if-thenist position. The first deals with the notion of structure and the second with the notion of consistency. I will take up the 'structure problem' in greater detail in Section 4 and the 'consistency problem' in Section 5; here I merely state the problems and give Resnik's solutions. Resnik first considers the objection that the structural option requires set theory as a background language since 'the concept of structure is usually defined in set-theoretic terms' [1980, p. 118]. But he is quick to remind us that the if-thenist taking this approach need not 'make any use of the existence assumptions of set theory'; so one may 'remain agnostic with respect to the existence of mathematical structures'. The benefits of such an agnostic, and what I have called,[9] *Carnapian approach* is that it 'gives mathematics a linguistic framework which is referential ... and thus agrees with the *prima facie* referential character of mathematical language as used by practicing mathematicians' [Resnik, 1980, p. 118].

Resnik next considers a list of problems related to the appeal to the notion of consistency. The first is the well-run vacuity problem; the second is that all inconsistent mathematical theories will define the same structure; the third is that to account for the depth and diversity of mathematics one must assume that the majority of mathematical theories are consistent. Faced with these problems, Resnik presents us with two alternative meta-mathematical routes. We can take the Fregean route of turning to philosophy and base the assumption of consistency on 'a belief in mathematical reality and truth which will vouchsafe the consistency of mathematical theories' [1980, p. 119]. The other way is to take a Carnapian route again of turning to logic and offer up a relative consistency proof to 'argue that since consistency is a mathematical question, it, too must be treated deductively [so] ... the assertion that a given axiom set is consistent must itself be construed as conditional upon a background theory with respect to whose truth the deductivist can remain agnostic' [1980, p. 119]. When confronted with both 'the structure problem' and 'the consistency problem', Resnik, Shapiro, Hellman, and mathematical foundationalists more generally, take the Fregean route. In contrast, I will opt for the Carnapian, but instead of turning to logical rules in an if-thenist context to justify our mathematical moves by treating relative consistency deductively, I will turn to mathematical method[10] in an as-ifist context to justify our

[9] See my [2003; 2006] where I argue that category theory is best understood as a Carnapian linguistic framework used to organize what we say about mathematical structures.

[10] My favoring methodological considerations over logical ones has a significant pluralist consequence in that no 'preferred' logic is presumed, and thus my Carnapian approach is intended as even more tolerant than Carnap's. That is, satisfiability may be expressed in

mathematical moves by treating relative consistency in terms of the notion of satisfiability. That is, what a relative consistency proof provides is an interpretation showing that the axioms are satisfiable, so that we can then get on with making structural as-ifist use of Putnam's [1979] semantic if-thenist schema.

Let's pause, then, to compare *metaphysically* interpreted structural *if-thenism* with *methodologically* interpreted structural *as-ifism* and see if we can't forestall these problems. Let's first recall that a basic premise of Plato's methodological as-ifism is that mathematics is used to solve *mathematical* and *physical* problems *and* that *it is in virtue of these uses* that we are justified in taking an object-level axiom *as if* it were *true*. Likewise, let's also presume that a basic premise of methodological structural as-ifism is that mathematics is also used to solve *meta-mathematical* problems *and* that it is in virtue of these uses that we are justified in taking a meta-level set of axioms *as if* they were *consistent*. Taking this methodological as-ifist route, by placing our focus on what is needed for the *practice* of mathematics, we are neither committed to the unconditional truth nor to the unconditional consistency of some background theory. Thus, in contrast to *metaphysical foundationalists*, what I will argue is that it is methodological considerations, and not metaphysical ones, that 'condition' the assumption of the truth of our object-level axioms or consistency of our meta-level axiom systems. What we get, then, is a version of structural as-ifism, much like Maddy's [2022] *enhanced* if-thenism. That is, we agree with Maddy that 'mathematics is a matter of figuring out what follows from what, where the concepts and axioms in the 'if' part are chosen with an eye to facilitating important mathematical goals' [Maddy, 2022, p. 277]. But, unlike Maddy, we do not analyze the 'what follows from what' in terms of a deductivist if-then reading; rather we analyze it in terms of a *structuralist as-if* reading,[11]

Recall that on Resnik's *if-then* structural perspective, we 'view a mathematical theory as studying the properties of all structures satisfying certain defining conditions, but never make use of the assumption that such structures exist'. Likewise, on my *as-if* structural account, we will adopt Putnam's *semantic*[12] if-thenist approach, whereby '*if* there is any structure that *satisfies* such-and-such axioms ... *then* that structure *satisfies* such-and-such further statements' [Putnam, 1979, p. 20, emphasis added], *but* we eschew reading the 'if...then' as a deductive 'if...then'[13] and instead read it as expressing the *methodological commitment* to our acting *as if* the axioms were first principles, *i.e.*, acting *as if* there is a structure that satisfies the axioms. So, our methodologically

any logic deemed suitable for solving the problem at hand. Thus, as with Shapiro [2014], I accept that there are a variety of logics at the mathematician's disposal.

[11] Maddy does consider Putnam's structural version of if-thenism, but claims that her deductive version seems 'to be a more natural rendition of the basic intuition that we're just trying to figure out "what follows from what" ' [Maddy, 2022, fn. 6].

[12] By 'semantic' I mean that this account is expressed in terms of satisfaction, not in proof-theoretic terms.

[13] Putnam too held that the 'if...then' was not a deductive one:

interpreted Putnamian schema becomes: when we act *as if* there is a structure that satisfies the axioms, then that structure satisfies such-and-such further statements. As with Plato, some of these commitments will be made with the goal of solving mathematical problems, some will be made with the goal of solving physical problems, and, too, some will be made with the goal of solving meta-mathematical problems.[14] *But* none of our commitments will be made with the goal of solving metaphysical problems, *i.e.*, problems concerning what 'fixes' the truth or the consistency of our axioms as first principles.

With respect to solving these meta-mathematical problems, does this mean that we will we have to call in model theory to solve the problem of what we mean by satisfaction? Yes, it does. But does this mean that, in our meta-mathematical determination of 'what follows from what', we need to take models themselves as possibly existing [Putnam, 1979] or as naturalistically constructed [Maddy, 2022]? No, it does not. With respect to framing what we mean by the concept of structure itself foundationally, will we have to call in some meta-mathematical linguistic framework? Yes, we will. Does that mean that we have to take structures themselves as actually [Shapiro, 1997] or possibly [Hellman, 1989] existing? No, it does not. Providing the details for these pronouncements will be the aim of the next sections. Before moving on to these arguments, however, I first consider mathematical structuralism itself and the various meta-mathematical languages used to frame it.

4. MATHEMATICAL STRUCTURALISM

As noted, the nineteenth-century view of mathematics saw the exactness of mathematics as lying in the precision of its definitions and the stability of the axiomatic method. As Burgess [2015] has beautifully detailed, the development of the axiomatic method, and the subsequent demand for rigor, brought with it a *structural* shift, that is, a shift that sees mathematics as, in the first instance, concerning itself with structure not with objects, so that mathematical objects are taken as nothing but positions in a structure. In light of this structuralist shift, I turn to reconsider, now from a structuralist perspective, our questions, *viz.*, Wherein lies the exactness of mathematics?; Wherein lie the object-level and meta-level conditions for speaking about mathematical structures? and, Wherein lies the exactness of the exact sciences?. I will then pause to remind

> Russell advocated a view of mathematics which he somewhat misleadingly expressed by the formula that mathematics consists of 'if-then' assertions. What he meant was not, of course, that all well-formed formulas in mathematics have a horseshoe as the main connective but that mathematicians are in the business of showing that *if* there is any structure which satisfies such-and-such axioms (*e.g.*, the axioms of group theory), *then* that structure satisfies such-and-such further statements (some theorems of group theory or other). [1979, p. 20]

But, in contrast to my methodological approach, Putnam [1979] went on to interpret this if-then in modal terms.

[14] It is these last meta-mathematical problems that will require a relative consistency proof to dissolve the various 'consistency problems'.

us that in mathematical and meta-mathematical and physical contexts we must be careful not to confuse the method of mathematics with the method of philosophy and so conflate mathematical considerations with metaphysical ones. More specifically, I will argue that philosophically-minded mathematical structuralists have continued to conflate metaphysical considerations with mathematical ones; that is, they assume that some *metaphysical background theory* is needed to account for mathematical axioms as first principles. Likewise, as I argue in my [2017] paper, in the context of philosophy of physics, philosophically-minded structural realists also conflate mathematical considerations with metaphysical ones; that is, they assume that set theory, category theory or group theory, taken as meta-level mathematical background theory, provides a metaphysical first-principled account of the priority of physical structure over physical objects. In contrast to such views, I argue for a methodological version of structural realism (MSR) that holds that we should only be realists about the mathematical structure that is needed to solve *object-level* physical problems, that is, problems that pertain to the empirical success of the theory.[15] My aim is to show that these conflations arise too from confusing the method of mathematics with the method of philosophy.

As noted in the previous section, for the mathematical structuralist, the exactness of mathematics is found in the axiomatic presentation of structure: axioms define structures (or define systems that have a structure), and objects are implicitly defined as positions in these structures (or in structured systems).[16] For example, the Peano–Dedekind axioms define the natural number structure, and a natural number is a position in any or all such structures. The

[15] In this sense, while my MSR interpretation is motivated by Plato's methodological approach it is also in line with recent interpretations of Aristotle's view (see, especially, [Lear, 1982] and [Franklin, 2014; 2021]) that it is an *empirical* claim to hold that the world itself has a given mathematical structure. In taking this Aristotelian stance, we avoid the metaphysically robust Pythagoreanism/Platonism that is adopted by Tegmark [2006] and too that threatens French's OSR. Thus, merging Plato's *methodological* aims with Aristotle's *empirical* ones, we arrive at the following account of the mathematical structure of the world: we develop our mathematical language with the aim of solving empirically motivated physical problems; we then further develop this language by our ascending to solving both mathematical and meta-mathematical problems, and only then are we in position to solve scientifically motivated physical problems by using mathematics to talk about the structure of the world. That is, to talk about the mathematical structure that explains *the empirical success* of our scientific theory and so justifies, via a no-miracles argument, that this is the structure of the world. So, we don't read the structure of the world from our mathematics as metaphysics, we read it from the empirical success of our mathematics as applied to the world. I thank an anonymous referee for pushing me to make more connections to Aristotle's account and to consider how this bears out against (structural realist) claims that the world has a mathematical structure.

[16] The use of the term 'structure' is typically associated with *ante rem* views (like Shapiro's [1997]) which hold that structure is something over and above any or all systems that have a structure, whereas the term 'structured system' is typically associated with *in re* views (like Hellman's [1989] and mine [2006; 2011]) which hold that there are only systems that have a structure. Since this issue is beside the point here, I will bypass it for now, and for ease of reading I will simply use the term 'structure'.

basic claim of the philosophically-motivated mathematical structuralist is that, in some sense, structure is *prior to* objects and so objects are *nothing but* positions in a structure. The structuralist position, then, is intended to supply us with an alternative to mathematical *metaphysical realism* by shifting the focus from talk of objects as prior, to talk of structure as prior. Further bringing these structuralist claims to the philosophical fore to argue against any set-theoretic realist reading of natural numbers as sets, is Benacerraf's [1965/1983] artful demonstration that natural numbers cannot be Fregean/set-theoretic, fixed-domain, objects. Thus, following this argument, we get the structuralist claim that the Peano–Dedekind axioms are prior and so natural numbers are nothing but positions in a structure or in any or all systems that are structured by the Peano–Dedekind axioms; likewise, groups are nothing but positions in a structure or in any or all systems that are structured by the group axioms, *etc.*

The mathematical-structuralist position can thus be summed up as follows: the *object-level* conditions for speaking about mathematical objects, like natural numbers themselves, groups themselves, are found in an axiomatically presented mathematical theory. At the object-level, when mathematical structuralists talk about numbers, groups, topological spaces, and yes, even sets and categories, we are *all* committed to the claim that structure, as defined axiomatically, is *prior to* objects, and so objects are *nothing but* positions in a structure. So at the object-level we act *as if* our axioms are true and it is this 'acting as if' that allows us to make methodological use of the Putnamian schema 'if there is any structure that satisfies such-and-such axioms ... then that structure satisfies such-and-such further statements', to get from the truth of the Peano–Dedekind axioms to the truth of, say, $2 + 2 = 4$, and from the truth of $2 + 2 = 4$ to the existence of 2 and 4 as positions in any or all systems satisfy these axioms.[17]

It is in working out the sense of priority at play that we are presented with our standard three philosophical interpretations of mathematical structuralism, *viz.*, we can take a *methodological*, an *ontological* or a *semantic* route to explaining what is meant by priority. For example, Reck's [2003] insightful reading of Dedekind's position reads his mathematical structuralism as a methodological position; *i.e.*, as a position that simply eschews both ontological and semantic interpretations of what priority might mean. To lay bare these various interpretations further in the context of current debates over 'the structure problem',

[17] So, much like Hilbert, we move from satisfiability to truth to existence. But note too that, unlike the realist who adopts an ontological route, we are not assuming that the reality of a mathematical structure, by itself, implies the reality of mathematical objects. We are assuming that we act as if our axioms are first principles so that the objects they talk about must exist. Likewise, against those who would hold that such as-ifism allows for non-realist interpretation which takes mathematical objects as 'posits', *i.e.*, as our axioms warranting our positing that they exist, I note that the as-ifist, taking as one does mathematical hypotheses as if they were true first principles, is committed to more than positing say 2 and 4 from the truth of the Peano–Dedekind axioms, since such mere positing would arise from our taking the axioms as mere hypotheses, as opposed to our taking them as if they were true first principles. I thank an anonymous referee for pushing me to mark these points of departure.

I next turn to consider the question: Wherein lie the *meta-level* conditions for speaking about structures or structured systems themselves? As we will see, it is in answering this meta-level question that the various present-day philosophical accounts of mathematical structuralism emerge.

Without going into all the details of the differences and similarities of these interpretations, and painting with a broad stroke, these options can be characterized as follows. The set-theoretic (ST) option takes objects as positions in a set structure or in any or all set-structured systems that have the same set structure, and so takes set theory as a *mathematical* background theory for speaking about structures or set-structured systems themselves. One can go one step further and argue for one specific set theory over another, and typically ZFC wins the day. The *ante rem* realist (AR) option of Shapiro [1997] *abstracts* from talk of systems and takes objects as abstract positions (places) in an *actual abstract* structure, and so takes structure theory as an *ontological* background theory for speaking about structures themselves. In contrast, the *in re* modal-nominalist (MN) approach of Hellman [1989], runs the other route, and *concretizes* talk of systems, taking objects as positions in any or all *possible* systems that have the same structure, and so takes nominalized modal logic as a *semantic* background theory for speaking of concrete systems themselves. Finally, the *in re* category-theoretic approach that I have advocated [2011] takes objects as positions in any or all systems that are cat-structured, where by cat-structured we mean *organized* by the EM, ETCS, or CCAF[18] axioms, and so takes category theory as a Carnapian background *language* (see my [1999]) for organizing what we say about structured systems themselves.

As explained in various papers (see especially [Landry, 2011; 2013]), my use of the term 'organizes' is meant to indicate that I am interpreting the category axioms in both a Hilbertian and Carnapian way; that is, I intend to take the category axioms as a 'schema' that is used to organize what we say about systems that have a structure, *without* providing a first-principled account of what structures or systems themselves are, in either an ontological or semantic sense of 'are'. But too, at both the object and the meta-level, the EM, ETCS, and CCAF axioms, as Hilbert-inspired structuralist schemata, are taken as prior in definition in so far as they implicitly define their objects as *nothing but* positions in any or all systems that have the same structure, and so, against metaphysical-realist interpretations, no objects need be taken as *prior in place*.[19] In this way, my as-if structuralism, much like that of Reck's Dedekind, is intended as a *methodological* position that eschews both ontological and semantic considerations. That is, we are to take the category theory *as if* it were a foundation, and so take the category axioms *as if* they were

[18] EM are the Eilenberg–Mac Lane axioms; ETCS are McLarty's elementary theory of category of sets axioms; and, CCAF are Lawvere's category of categories as a foundation axioms.

[19] See Aristotle's *Metaphysics, Books M and N* for more on the distinction between prior in definition and prior in place, and for his (mistaken) claim that Plato held that mathematical objects were prior in place, whereas he took them as prior in definition.

first principles, and this with the aim of answering the question: Wherein lie the meta-level conditions for speaking about structures or structured systems themselves?

But, and as distinct from anything I have written on this topic, this position in now presented as a Plato-inspired *as-ifist methodological* position, *viz.*, it is for purely *methodological* reasons, at both the object and the meta-level, that we act *as if* the axioms of a theory were first principles, but too we know that they are but hypotheses that we make for the purpose of solving problems. And it is just because we act *as if* the category axioms are first principles, all the while knowing that they are not, that we eschew both ontological and semantic considerations, *i.e.*, considerations of what would 'make them' or 'fix them' as first principles. On this view, the *object-level* conditions for speaking about mathematical objects themselves as nothing but positions in a structured system are found in an axiomatically presented mathematical theory, which itself precisely defines its objects. Mathematical objects themselves are thus to be taken at face value, *i.e.*, groups, rings, sets, topological spaces, categories are just what the group, ring, set, category, *etc.*, axioms say they are. That is, at the object level the set, group, category, *etc.*, axioms are taken *as if* they were true first principles, but they are not; simply, they are hypotheses we set up to give us those objects that we need to solve mathematical problems or physical problems.

Next, and akin to the double role played by the theory of proportion in Plato's account, category theory is further used to answer the question: Wherein lie the *meta-level* conditions for speaking about structured systems themselves? That is, in line with my methodological reading of Plato's as-ifism, Reck's methodological reading of Dedekind, and Resnik's Carnapian reading of set theory, category theory is to be taken as a *methodological metalanguage* used to organize or frame what we say about structured systems themselves, *e.g.*, systems like **Set**, **Grp, Top** and yes, even **Cat**.[20] That is, the EM axioms are used to talk about sets, groups, topological spaces and categories themselves as cat-structured systems; the ETCS axioms are used to talk about set-structured systems; and, the CCAF axioms are used to talk about cat-structured systems, *i.e.*, categories themselves. The category axioms, again as Hilbertian schemata, are taken to define structured systems themselves implicitly as nothing but positions in any or all systems that have the same structure, and as such they are taken as *prior in definition*.

Thus, against any metaphysical realist position, categories themselves need not, in *any* sense, be taken as *prior in place*. So considered, there is no need to presume a fixed domain for either reference or meaning; so, against [Hellman,

[20] **Set** the category of (small) sets, **Grp** the category of (small) groups, **Top** the category of (small) topological spaces, **Cat** the category of (small) categories. Lest one be tempted here to sneak in some large cardinal set theory as a metaphysical background theory in attempting to solve the problem of 'larger size' categories, rest assured that one can use Gödel–Bernays (GB) class theory or Grothendieck universes as a background language.

2003], there are simply no ontological or semantic 'home address' problems to face. More to the point, the EM, ETCS, and CCAF axioms are taken *as if* they were true first principles, but they are not; they are hypotheses that we set up for the *methodological* purpose of solving mathematical problems and, for the philosophical structuralist, one of these problems is the meta-level question of how to talk about structured systems themselves. Thus, my answer to Hellman's 'home address problem', or equally, to 'the structure problem' as applied to categories themselves, is that we take the CCAF axioms as a *Carnapian* background *language* for organizing what we say about cat-structured systems themselves, *i.e.*, for what we say about categories themselves as objects.

5. MATHEMATICS IS NOT METAPHYSICS

It is these differences, of taking our meta-mathematical axioms as both linguistic and *as if* they were true first principles, that distinguish my *methodologically* interpreted structural as-ifist position from the above considered foundationalist, or *metaphysical*, options. For example, the approaches of ST, AR, and MN, (see Section 4) all see their axioms as *asserting* truths over a *fixed domain*,[21] and so problematically force us to take their proposed background theory as a *metaphysical foundation* that gives a first-principled account of structures or structured systems by fixing reference or meaning to sets, to actual abstract structures, or to possible concrete systems. This has the concerning consequence that taking ST, AR, or MN as a background theory of structures or systems commits us to the meta-level *metaphysical realist* claim that sets, actual structures, or possible systems must be taken as in some sense *prior in place* and, as such, that ontological or semantic conditions are needed to account for what priority might mean here, *e.g.*, what we might mean by ontologically abstract or modally concrete. Yet another problematic result of such metaphysical foundational approaches is that, at the object level, mathematical objects, like numbers, vectors, groups, topological spaces, *etc.*, are no longer taken at face value; that is, they are taken as 'reducible' to sets, or as 'instances' of actual structures, or as 'expressible' in terms of possible concrete systems.

Now one might well claim that what this shows is that a pluralist position with respect to mathematical structuralism is possible and that, especially as Carnap-inspired philosophers of mathematics, we should simply let the many flowers bloom where they may. But what I will argue next is that it is just these metaphysically motivated foundationalist flowers that push us to a view of mathematics that is not amenable to the practice or the applicability of mathematics. And the reason for the disconnect is that same: these views confuse the first-principle/domain-fixing method of philosophy with the hypothesis/definition-fixing method of mathematics and, in so doing, conflate metaphysical considerations with mathematical ones. For example, for the ST,

[21] Shapiro's [1997] account takes axioms as asserting ontological truths about actual structures, whereas Hellman's [1989] account takes axioms as asserting modal truths, more precisely as he takes them as the antecedents of model truths that range over possible systems.

AR, and MN options these first-principled conditions are taken to lie in some *metaphysical theory* that aims to fix what structures, actual structures, or possible systems 'are', in either an ontological or a semantic sense.

The ST advocate, can, however, avoid this metaphysical route and, like the CT option, argue that the needed as-if conditions are taken to lie in some *meta-mathematical theory* that, *for methodological purposes*, is taken to define implicitly structures or systems. Thus, on either interpretation, the CT or ST axiom systems may be taken as a linguistic framework that we use to organize what we say about systems that have a structure, without our having to turn to metaphysics, that is, without our having to reify our talk of either set-structures or structures themselves, or nominalize our talk of structured systems themselves.

So why, then, do I advocate taking CT as *the* linguistic framework for mathematical structuralism? Here are my reasons:

1. We can give an as-if methodological and schematic reading of ST, but will still have a fixed domain (of sets for ZFC, classes for GB (fn. 20), urelements for ZFA) as either ontologically or semantically prior; for CT, the axioms need only be taken as definitionally prior.
2. CT takes the objects of mathematics at face value; because 'objects' and 'arrows' are taken, as with Mac Lane, as 'undefined terms or predicates' [Mac Lane, 1968, p. 287], the objects it talks about, *e.g.*, sets, groups, topological spaces, deductive systems, can be taken at face value and so do not require a 'reduction to' set structure, again, in the sense of either an ontological or semantic reduction.
3. CT does better at capturing the shared structure of the various types of structured systems in terms of functors, identity maps, category equivalence, *etc.*, that is, we are not restricted merely to isomorphism and the many problems this notion brings.[22]
4. This position allows us to see better that, even if we use ST or CT to characterize the structure of scientific theories, the *applicability* of mathematical structure in science that does the *empirical work* for the structural realist is at the object level, not the meta-level. This then allows us to appreciate better that the exactness of the exact sciences lies in the application of object-level and *not* meta-level mathematical structure. So that even if the structure of scientific theories is framed by category theory or by set theory, this, in itself, tells us nothing about the structure of the world.

Thus, I come to the conclusion that I have argued for in many papers but now with an as-ifist twist: using CT as a background *metalanguage* for mathematical structuralism shows that it is possible to speak *as if* structure were

[22] See Marquis' [2020] excellent paper detailing the development of these notions, from Bourbaki's set-theoretic account to those currently given in the context of category theory.

prior to objects and *as if* objects were *nothing but* positions in a system that has a structure, *without* our having to presume CT, or indeed any theory, as a background *metaphysical* foundation.

Again, and to sum up the arguments presented in this paper: at the *object level*, when *mathematical* structuralists talk about numbers, groups, and yes, even sets and categories, we are *all* committed to the claim that structure is *prior to* objects, and so these objects are *nothing but* positions in a structure. The *philosophical* structuralists' *meta-level* problem is then set at working out the sense of priority at play here. To this end, we are presented with three philosophical ways of resolving this 'priority problem', *viz.*, we can take an *ontological*, a *semantic*, or a *methodological* route. I have argued that both the ontological and the semantic routes are ways not to be looked into because they take their meta-mathematical axioms as first principles and, in so doing, they conflate the metaphysical method of philosophy with the hypothetical method of mathematics by presuming that the sense of priority at play is a metaphysical one, *i.e.*, that it depends on some ontological or semantic objects (sets, actual structures, possible systems) being presumed as *prior in place*. The methodological route, in contrast, reverses this and instead argues that it is in virtue of solving the meta-mathematical structuralist's 'structure problem' that we are justified in taking the CT axioms *as if* they were first principles, all the while realizing that they are not; so that the only notion of priority presumed to be at play for the schematic terms 'object', 'arrow', and 'category' is that they are *prior in definition*.

To press this point further, I now want to combine our two lessons, one from Carnap *and* one from Plato, *viz.*, that we reject background *metaphysical theories* in favor of a background *language and* that we do so, not in virtue of logical considerations as Carnap suggests or in virtue of metaphysical considerations as foundationalists have done, but rather, as both Plato and Maddy [2022] suggest, in virtue of methodological considerations as set in the context of the *practice* of mathematics, that is, in the context of solving mathematical, meta-mathematical, and physical problems. As Resnik [1980] has already pointed out for the structural (as opposed to the deductive) if-thenist, even if we agree that the *structural as-if* option requires some theory as a meta-level background language to 'define the concept of structure', the as-ifist need not 'make any use of the existence assumptions' of the background language, so that one may 'remain agnostic with respect to the existence of mathematical structures' [1980, p. 118]. Recall too that the benefits of such a Carnapian approach is that it 'gives mathematics a linguistic framework which is referential ... and thus agrees with the *prima facie* referential character of mathematical language as *used by practicing mathematicians*' [1980, p. 118, emphasis added]. Having thus set the stage for my methodologically motivated meta-mathematical category-theoretic structural as-ifism, let's now get clearer about what exactly is at issue when I, likewise, claim that much of the current philosophical structuralist debate about 'the consistency problem' has also arisen from a conflation of the method of philosophy with the method of mathematics.

To this end, let's now consider Shapiro's [2005] claim that, at the meta-mathematical level, when faced with the 'consistency problem', the CT option collapses; its advocates must either accept one of the ST, AR, or MN options outlined above, or they must turn to logic or turn to philosophy. I first remind the reader that my methodologically motivated meta-mathematical category-theoretic structural as-ifism is used to answer the philosopher's *meta-mathematical* question: Wherein lie the *meta-level* conditions for speaking about structured systems themselves? Now to reply to Shapiro, I note that it is in virtue of this use that we are *philosophically justified* in taking the category axioms *as if* they were *consistent*. Taking the methodological as-ifist route, we are neither committed to the unconditional consistency of our CT axioms nor to the unconditional truth of CT as a background theory. What, then, conditions, or *mathematically justifies*, our meta-mathematical claim that we are to treat our CT axioms as if they were consistent? Recalling our Putnamian *as-if* semantic schema, we want this to be answered in terms of satisfiability. And so, as both Resnik [1980] and Shapiro [2005, p. 74] note, to justify the as-if claim of the consistency of our axioms systems mathematically all we need be committed to is the existence of a relative consistency proof demonstrating the satisfiability of our axioms.

But Shapiro, again conflating the method of philosophy with the method of mathematics, is quick to counter that when it comes to the statement of consistency itself, we yet need be committed to the unconditional truth of some background theory. That is, if we are to avoid the assumed 'ensuing vicious regress' of relative consistency proofs, then we must accept that 'the meta-theory — the mathematical theory in which the consistency of an axiomatization is established — is not to be understood algebraically, not as another theory of whatever satisfies *its* axioms' [2005, p. 70]. Failing this, we must either turn to logic and, for example, appeal to a completeness theorem, which of course would require a first-order formalization and again a proof in some background theory,[23] or we must turn to philosophy and concede that consistency itself is not a logical notion. Our only other way out of this 'vicious regress', according to Shapiro [2005, p. 70], is to require an *unconditionally true* 'assertory theory of sets or structures to meta-mathematically account for the

[23] In the context of the objection that the category-theoretic foundationalist will have to prove the completeness theorem in some background theory, Hellman notes that any attempt to avoid this objection by taking the category axioms as hypothetical, will result in 'the old "if-then-ist" predicament that plagues deductivism: what we thought we were establishing as determinate truths turn out to be merely hypothetical, dependent on the mathematical existence of the very structures we thought we were investigating, and threatening to strip mathematics of any distinctive content' [Hellman, 2003, p. 138]. Likewise, in his most recent [2021] paper, Hellman argues that any if-thenist account, including Maddy's [2022] enhanced if-thenism, will fall victim to his 'loss of structural content' objection because it takes axioms as bare antecedents in purely logical conditional assertions as opposed to its needing to take them as true or possibly true assertions. Note, however, that my as-ifist view avoids both of these objections because it takes axioms not as merely hypothetical but *as if* they were true.

needed notion of satisfiability'. But note that on my methodological account, to solve this meta-mathematical problem of the regress of relative consistency proofs in model-theoretic terms, we need only act *as if*, for example, set theory were true, and, even then, we need only act as if it were true *for this purpose*. To think otherwise is again to conflate the hypothetical method of mathematics with the metaphysical method of philosophy. Simply put, if anyone is 'turning to philosophy', it is Shapiro and other metaphysical foundationalists!

I repeat, with respect to solving these meta-mathematical problems, does this mean that we will we have to call in model theory to solve the problem of what we mean by satisfaction? Yes, it does. Does that mean that we will have to take models as possibly existing or as naturalistically constructed? No, it does not. With respect to foundationally framing what we mean by the concept of structure, will we have to call in some background linguistic framework? Yes, we will. Does that mean that we have to take structures as set-structured, or as actually or possibility existing? No, it does not. That is, in contrast to ST, AR, and MN foundationalists, it is methodological considerations, and not metaphysical ones, that condition the as-if assumption of the relative consistency of our meta-level axioms. Thus, when answering the philosophers' question: Wherein lie the *meta-level* conditions for speaking about structured systems themselves?, we are committed to taking our CT axioms *as if* they were consistent; this allows us methodologically to act *as if* category theory were a foundation for mathematical structuralism. Yet, as with Plato, we all the while realize that, metaphysically speaking, it is not!

My claim can now be precisely put as follows: arguments for foundationalist metaphysical approaches, as contrasted against my proposed as-if foundationalist methodological approach, hinge on the mistaken belief that the method of mathematics must match the method of philosophy so that at least some meta-mathematical axioms must be taken as first principles and that the acceptability of these (their truth, consistency, *etc.*) must be pre-established by their being tethered to a fixed domain of stable objects, be these sets, actual structures, or possible systems. This is an error; as I hope I have shown; the exactness of mathematics, unlike the exactness of philosophy, is founded on the precision of its definitions and the stability of its method. It is not founded on the stability of any objects, be their stability ontologically or semantically fixed. Rather, the exactness of mathematics is founded on our taking both our object-level and our meta-level mathematical axioms as hypotheses but acting *as if* they were first principles for the purpose of solving mathematical, foundational, and physical problems.

Moreover it is this *methodological structural as-ifism* that allows us to capture the object-level if-thenism of Putnam's structuralist if-thenist, wherein we may claim that '*if* there is any structure that satisfies such-and-such axioms ... *then* that structure satisfies such-and-such further statements' [Putnam, 1979, p. 20], but we eschew reading the 'if... then' as a *logical* 'if... then' and instead read it as expressing the *methodological* commitment to act *as if* the mathematical axioms were first-principles so that we can claim that there is indeed such a structure that satisfies the axioms. And finally, with both Plato and

Maddy, we acknowledge that some of those commitments will be made with the goal of solving mathematical problems and too some will be made with the goal of solving *physical* problems. As a consequence, this methodologically motivated structural as-ifist account of the *applicability* of mathematics will be in line with Plato's account of the applicability of pure mathematics in astronomy and cosmology, Aristotle's account that sees any claim about the structure of the world as a claim that must be justified empirically, and Maddy's account of the applicability of pure mathematics in physics. As regards the claims of structural realists, we have again carved out a methodological midpoint: claims of the structure of the world will reside in the claim that the exactness of the exact sciences lies in the *applicability* of *pure* mathematics; they will not lie either in any empirically motivated mathematics or in any metaphysically interpreted mathematics. This has the result that applicability is, as Maddy [2022, pp. 270–271] states, 'the claim that this abstract model [read now as a structured system] resembles the worldly situation well enough to be used for certain purposes', but, as with Aristotle, these purposes must be measured empirically not metaphysically!

Thus, and to conclude, my overall lesson is this: when we shift our focus from the method of philosophy to the method of mathematics, we see that an *as-if methodological* interpretation of mathematical structuralism can be used to provide an account of the practice of and the applicability of mathematics while avoiding the conflation of metaphysical considerations with mathematical ones.

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